Sudden death and long-living of entanglement in an ion-trap laser

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The dynamical properties of quantum entanglement in a time-dependent three-level-trapped ion interacting with a laser field are studied in terms of the reduced-density linear entropy considering two specific initial states of the field. Allowing the instantaneous position of the center-of-mass motion of the ion to be explicitly time-dependent, it is shown that either sudden death of entanglement or survivability of quantum entanglement can be obtained with a specific choice of the initial state parameters. The difference in evolution picture corresponding to the multi-quanta processes is discussed.

1 Introduction

One of the most striking differences between classical and quantum correlations is the restricted capability of quantum states to share entanglement. The decay of entanglement cannot be restored by local operations and classical communications, which is one of the main obstacles to achieve the quantum computer [1]. Therefore it becomes an important subject to study the loss of entanglement [2, 3, 7, 8, 9, 10]. Quite recently, by using vacuum noise two-qubit, entanglement is terminated abruptly in a finite time have been performed [2] and the entanglement dynamics of a two two-level atoms model have been discussed [3]. They called the non-smooth finite-time decay entanglement sudden death.

On the other hand, trapped atomic ions are an ideal system for exploring quantum information science. Recent advances in the dynamics of trapped ions (for a recent review, see e.g., [4]) have demonstrated that a macroscopic observer can effectively control dynamics as well as perform a complete measurement of states of microscopic quantum systems. With the reliance in the processing of quantum information on a cold trapped ion, a long-living entanglement in the ion-field interaction with pair cat states has been observed [5] . Also, experimental preparation and measurement of the the motional state of a trapped ion, which has been initially laser cooled to the zero-point of motion, has been reported in [6].

In this paper we present an explicit connection between the initial state setting of the field and the dynamics of the entanglement. We give a condition for the existence of either entanglement sudden death or long-lived entanglement. In particular, a quantitative characterization of a general system of a three-level trapped ion interacting with a laser field is presented. We present various numerical examples in order to monitor the linear entropy and entanglement

dynamics. The paper is organized as follows. In section 2, we consider a general class of a three-level system and obtain its solution. In section 3 we discuss the dynamics of the entanglement with different initial states. Finally, we summarize the results and conclude in section 4.

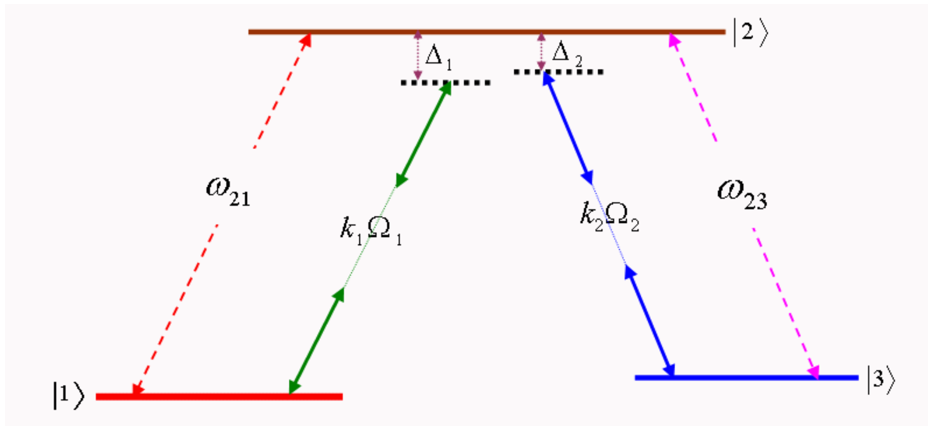

Figure 1: Energy-level diagram for a three-state $\Lambda-$type system interacting with a bimodal cavity field coupling the two ground states $|1\rangle$ and $|3\rangle$ to a common excited state $|2\rangle$ via a Raman transition.

2 Model

Now let us consider the Hamiltonian which describes a single trapped ion in a two-dimensional trap. Therefore, the physical system on which we focus is a three-level harmonically trapped ion with its center-of-mass motion quantized. We denote by $\hat{\psi}_i$ and $\hat{\psi}_i^{\dagger}$ the annihilation and creation operators and $\upsilon_1(\upsilon_2)$ is the vibrational frequency related to the center-of-mass harmonic motion along the direction $\hat{x}(\widehat{y})$. The trapped ion Hamiltonian may be written as [11, 12, 13]

$$\begin{aligned}
\hat{H} &= \hat{H}_0 + \hat{H}_{int}(t), \\
\hat{H}_0 &= \hbar \sum_{i=1}^{2} \upsilon_i \hat{\psi}_i^{\dagger} \hat{\psi}_i + \hbar \sum_{i=1}^{3} \omega_i \hat{S}_{ii}, \\
\hat{H}_{int}(t) &= \hbar \Im_1(\hat{x}, t) \hat{S}_{12} + \hbar \Im \lambda_2 \Im_2(\widehat{y}, t) \hat{S}_{31} + \hbar \Im_1^*(\hat{x}, t) \hat{S}_{21} + \hbar \Im_2^*(\widehat{y}, t) \hat{S}_{13}. \qquad (1)
\end{aligned}$$

We denote by $\hat{S}_{lm}$ the atomic flip operator for the $|m\rangle \rightarrow |l\rangle$ transition between the two electronic states, where $\hat{S}_{lm} = |l\rangle\langle m|, (l, m = 1, 2, 3)$.

So far we have disregarded relaxations since we are interested in the dynamics for short times. Suppose the ion is irradiated by a laser field of the form

$$\begin{aligned}\Im_1(\hat{x},t) &= \frac{\epsilon_1\langle 1|d_1.\wp_1|2\rangle}{\hbar}\exp[-i(k_1\hat{x}-\Omega_1 t)],\\ \Im_2(\hat{y},t) &= \frac{\epsilon_2\langle 1|d_2.\wp_2|2\rangle}{\hbar}\exp[-i(k_1\hat{y}-\Omega_2 t)],\end{aligned} \tag{2}$$

where ϵ_1 and ϵ_2 are the amplitudes of the two laser fields with frequencies Ω_1 and Ω_2 and polarization vectors $\wp_1$ and $\wp_2$, respectively. The transition in the three-level ion is characterized by the dipole moment d_i and $k_i, i = 1, 2$ are the wave vectors of the two laser fields.We define the detuning between the atomic transitions and the fields as $\Delta_1 = \omega_{21} - m_1\Omega_1$ and $\Delta_2 = \omega_{23} - m_2\Omega_2$.

Therefore if we express the center of mass position in terms of the creation and annihilation operators of the two-dimensional trap namely

$$\hat{x} = \Delta x(\hat{\psi}_1^\dagger + \hat{\psi}_1), \qquad \text{and} \qquad \hat{y} = \Delta y(\hat{\psi}_2^\dagger + \hat{\psi}_2). \tag{3}$$

where $\Delta x = (\hbar/2\upsilon_1 m)^{1/2} = \eta_1/k_1$ and $\Delta y = (\hbar/2\upsilon_2 m)^{1/2} = \eta_2/k_2$ are the widths of the bi-dimensional potential ground states, in the x and y directions (η_i is called Lamb–Dicke parameter describing the localization of the spatial extension of the center-of-mass in $i^{\underline{th}}$ direction), and m is the mass of the ion.

Making use of the special form of Baker-Hausdorff theorem [14] the operator $\exp[i\eta(\hat{\psi}_1^\dagger+\hat{\psi}_1)]$ may be written as a product of operators i.e. $\exp(i\eta(\hat{\psi}^\dagger+\hat{\psi})) = \exp\left(\frac{\eta^2}{2}[\hat{\psi}^\dagger,\hat{\psi}]\right)\exp\left(i\eta\hat{\psi}^\dagger\right)\exp\left(i\eta\hat{\psi}\right)$. The physical processes implied by the various terms of the operator

$$\exp\left(i\eta\left(\hat{\psi}^\dagger+\hat{\psi}\right)\right) = \exp\left(\frac{-\eta^2}{2}\right)\sum_{n=0}^{\infty}\frac{(i\eta)^n\,\hat{\psi}^{\dagger n}}{n!}\sum_{m=0}^{\infty}\frac{(i\eta)^m\,\hat{\psi}^m}{m!}. \tag{4}$$

may be divided into three categories (i) the terms for $n > m$ correspond to an increase in energy linked with the motional state of center of mass of the ion by $(n-m)$ quanta, (ii) the terms with $n < m$ represent destruction of $(m - n)$ quanta of energy thus reducing the amount of energy linked with the center of mass motion and (iii) $(n = m)$, represents the diagonal contributions. When we take Lamb-Dicke limit and apply the rotating wave approximation discarding the rapidly oscillating terms, the effective interaction Hamiltonian (1) takes the form

$$\begin{aligned}\hat{H}_{int} &= \hbar\gamma_1(t)\mathcal{E}_p^{(1)}(\hat{\psi}_1^\dagger\hat{\psi}_1)\hat{S}_{12}\hat{\psi}_1^{\dagger m_1} + \hbar\gamma_2(t)\mathcal{E}_p^{(2)}(\hat{\psi}_2^\dagger\hat{\psi}_2)\hat{S}_{23}\hat{\psi}_2^{\dagger m_2} + \hbar\gamma_1^*(t)\mathcal{E}_p^{(1)*}(\hat{\psi}_1^\dagger\hat{\psi}_1)\hat{S}_{21}\hat{\psi}_1^{m_1}\\ &\quad +\hbar\gamma_2^*(t)\mathcal{E}_p^{(2)*}(\hat{\psi}_2^\dagger\hat{\psi}_2)\hat{S}_{32}\hat{\psi}_2^{m_2},\end{aligned} \tag{5}$$

where $\gamma_i(t)$ is a new coupling parameter adjusted to be time dependent. The other contributions are rapidly oscillating with frequency ν and have been disregarded. Note that in the Lamb-Dicke regime only processes with $p = 0, 1$ are considered, while in the general case, the nonlinear coupling function is derived by expanding the operator-valued mode function as

$$\mathcal{E}_k^{(j)}(\hat{\psi}_i^\dagger \hat{\psi}_i) = -\frac{\epsilon_i}{2} \exp\left(-\frac{\eta_i^2}{2}\right) \sum_{n=0}^{\infty} \frac{(i\eta_i)^{2n_i+k}}{n_i!(n_i+k)!} \hat{\psi}_i^{\dagger n} \hat{\psi}_i^n. \tag{6}$$

Since $\mathcal{E}_k^{(j)}(\hat{\psi}_i^\dagger \hat{\psi}_i)$ depends only on the quantum number $\hat{\psi}_i^\dagger \hat{\psi}_i$, in the basis of its eigenstates, $\hat{\psi}_i^\dagger \hat{\psi}_i |n_i\rangle = n_i |n_i\rangle$, $(n = 0, 1, 2, ...)$, these operators are diagonal, with their diagonal elements $\langle n|\mathcal{E}_k^{(j)}(\hat{\psi}_i^\dagger \hat{\psi}_i)|n\rangle$ is given by $\mathcal{E}_k^{(j)}(n_i) = -0.5\epsilon_i((n_i+k)!)^{-1} n_i! L_{n_i}^k(\eta_i^2) \exp(-\eta_i^2/2)$ where $L_{n_i}^k(\eta_i^2)$ are the associated Laguerre polynomials.

In what follows we obtain a general result regarding the solution to the time evolution operator. Now, we expand the time evolution operator in terms of the complete set of atomic operators as

$$\hat{U}(t) = \exp\left(-i \int_0^t \hat{H}_{int}(\tau) d\tau\right). \tag{7}$$

In the basis of the eigenstates, $|n_1, n_2\rangle$, we can find the elements of $\hat{U}(t)$ as $\mathfrak{S}_{ii}(n_1, n_2, t) = \langle i|\hat{U}(n_1, n_2, t)|j\rangle$, where $\hat{U}(n_1, n_2, t) = \langle n_1, n_2|\hat{U}(t)|n_1, n_2\rangle$. Let us discuss the problem under consideration with $\varepsilon_i = \varepsilon$, $\phi_j = 0$, and the initial conditions $\mathfrak{S}_{ii}(n_1, n_2, 0) = 1$ and $\mathfrak{S}_{ij}(n_1, n_2, 0) = 0$, $(i \neq j)$. Under these conditions and after straightforward calculations, one can find an analytic time dependent solution in the following forms

$$\begin{aligned}
\mathfrak{S}_{11}(n_1, n_2, t) &= \frac{1}{\mu_{n_1,n_2}^2} \left\{ \frac{\gamma_1^2 (n_1+m_1)!}{n_1!} \cos\left(\mu_{n_1,n_2} \int_0^t \gamma(\tau) d\tau\right) + \frac{\gamma_2^2 (n_2+m_2)!}{n_2!} \right\}, \\
\mathfrak{S}_{22}(n_1, n_2, t) &= \cos\left(\mu_{n_1,n_2} \int_0^t \gamma(\tau) d\tau\right), \\
\mathfrak{S}_{33}(n_1, n_2, t) &= \frac{1}{\mu_{n_1,n_2}^2} \left\{ \frac{\gamma_2^2 (n_2+m_2)!}{n_2!} \cos\left(\mu_{n_1,n_2} \int_0^t \gamma(\tau) d\tau\right) + \frac{\gamma_1^2 (n_1+m_1)!}{n_1!} \right\}, \\
\mathfrak{S}_{12}(n_1, n_2, t) &= \frac{-i\gamma_1}{\mu_{n_1,n_2}} \sqrt{\frac{(n_1+m_1)!}{n_1!}} \sin\left(\mu_{n_1,n_2} \int_0^t \gamma(\tau) d\tau\right), \\
\mathfrak{S}_{13}(n_1, n_2, t) &= \frac{\gamma_1 \gamma_2}{\mu_{n_1,n_2}^2} \left\{ \cos\left(\mu_{n_1,n_2} \int_0^t \gamma(\tau) d\tau\right) - 1 \right\}, \\
\mathfrak{S}_{23}(n_1, n_2, t) &= -\frac{i\gamma_2^*}{\mu_{n_1,n_2}} \sqrt{\frac{(n_2+m_2)!}{n_2!}} \sin\left(\mu_{n_1,n_2} \int_0^t \gamma(\tau) d\tau\right).
\end{aligned} \tag{8}$$

The rest of $\mathfrak{S}_{ij}(n_1, n_2, t) = (\mathfrak{S}_{ji}(n_1, n_2, t))^*$ and the generalized Rabi frequency μ_{n_1,n_2} is given

by

$$\mu_{n_1,n_2} = \sqrt{\frac{\gamma_1^2\mathcal{E}_1^2(n_1)(n_1+m_1)!}{n_1!} + \frac{\gamma_2^2\mathcal{E}_2^2(n_2)(n_2+m_2)!}{n_2!}}. \tag{9}$$

We have thus completely determined the exact solution of a three-level trapped ion in the presence of time-dependent modulated function. Let's observe that putting in particular $m_i = 1$, $\gamma(\tau) = 1$ and looking at the present solution (8), we recover the time-independent three-level system discussed in [11, 12].

The (internal level) ionic dynamics depend on the distributions of initial excitations of both the field and the center-of-mass vibrational motion, given by $\rho_F(0)$ and $\rho_A(0)$, respectively. For instance, through a unitary evolution operator, the final state $\rho(t)$ can be calculated in the following expression

$$\rho(t) = \hat{U}(t)\left(\rho_A(0) \otimes \rho_F(0)\right) U^{\dagger}(t). \tag{10}$$

Having obtained the explicit form of the final state of the system $\rho(t)$, we can discuss any statistical property of the system. We will turn our attention to the time evolution of linear entropy and entanglement, when the field is initially in the Fock state or coherent state.

3 Entanglement dynamics

The entanglement can be described by the linear entropy or the von-Neumann entropy [15]. The most prominent choice of pure state entanglement measures is the von-Neumann entropy [15, 16, 17, 18]

$$S\left(\rho_{A(F)}\right) = -tr\left(\rho_{A(F)} \ln \rho_{A(F)}\right),$$

of the reduced density matrix, often simply called the entanglement $E(\psi) = S\left(\rho_{A(F)}\right)$ of the pure state $|\psi\rangle$. We work with the linear entropy which is convenient to calculate [19], which is given by

$$S_A(t) = 1 - tr_A\left(\rho_A^2(t)\right), \tag{11}$$

which ranges from 0 for a pure state to 1 for a maximally entangled state and tr_A denotes the trace over the subsystem A. The linear entropy is generally a simpler quantity to calculate than the von Neumann entropy as there is no need for diagonalization and can be considered as a very useful operational measure of the atomic state purity.

In figure 2, numerical results for the time evolution of the linear entropy for an initial Fock state of the fields ($|n, m\rangle$, with $n = m = 0$) have been presented. In the typical experiments at NIST [20], a single $^9Be^+$ ion is stored in a RF Paul trap with a secular frequency along $\widehat{x}$ of $\nu/2\pi \simeq 11.2$ MHz, providing a spread of the ground state wave function of $\Delta x \simeq 7$ nm,

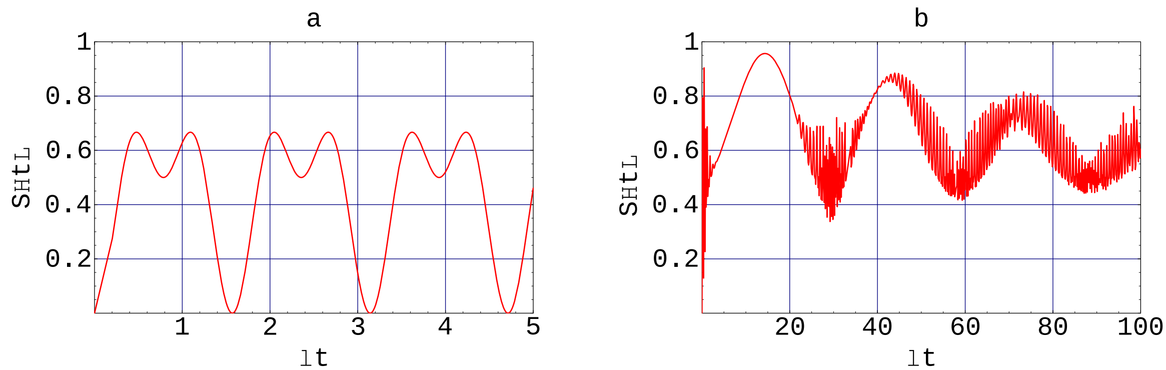

Figure 2: Example of the numerical calculation of $S_A(t)$ as a function of the scaled time λt, $(\lambda = \gamma_1 = \gamma_2)$. The parameters $m_i = 1$, $\gamma(\tau) = 1$ and different initial states of the field, where, (a) Fock state with $(n_i = 0)$ and (b) coherent state with $(\overline{n}_i = 0)$.

with a Lamb-Dicke parameter of $\eta \simeq 0.202$. The two laser beams, with 0.5 W in each one, are approximately detuned $\Delta/2\pi \simeq 12$ GHz, so that $\gamma_i/2\pi \simeq 475$ kHz. With these data we find $\epsilon_i \simeq 0.01$, so they can be considered as small parameters. The case of the effective vacuum is quite interesting where the linear entropy oscillates between zeros and a maximum value, in this case $S_A(t) \simeq 0.65$ (see figure 2a). In fact the linear entropy attains the zero value (i.e., disentanglement) when the trapped ion is either in its upper or lower states (i.e., pure state) while strong entanglement occurs when the inversion is equal to zero. On other words, due to initial Fock state, the entanglement reaches its maximum value and drops to zero periodically, which opens the door for a possible application of the present model in constructing a quantum logic gate.

The immediate question now is, if different initial states of the field are considered, is the periodic behavior of the linear entropy and zeros entanglement for such states still exist? To answer this question we make use of a coherent state as an initial state of the field and find a general entanglement feature, captured in equation (11) and illustrated in figure 2b. Once the initial state of the field is considered to be a coherent state the situation is changed drastically (see figure 2b). It is obvious that the time evolution of linear entropy behaves as that of standard single-photon Jaynes-Cummings model and oscillates irregularly with the time. At the early times the linear entropy from zero evolves to its local maximum value ($\simeq 0.96$). In this process, the three level-trapped ion and the fields are always entangled. Although increasing the mean photon number leads to strong entanglement (maximum value of entanglement), however the maximum value of the entanglement also varies and occurs for some short period of time. This indicates that in a regime where coherent state is considered, the underlying states are highly

entangled.

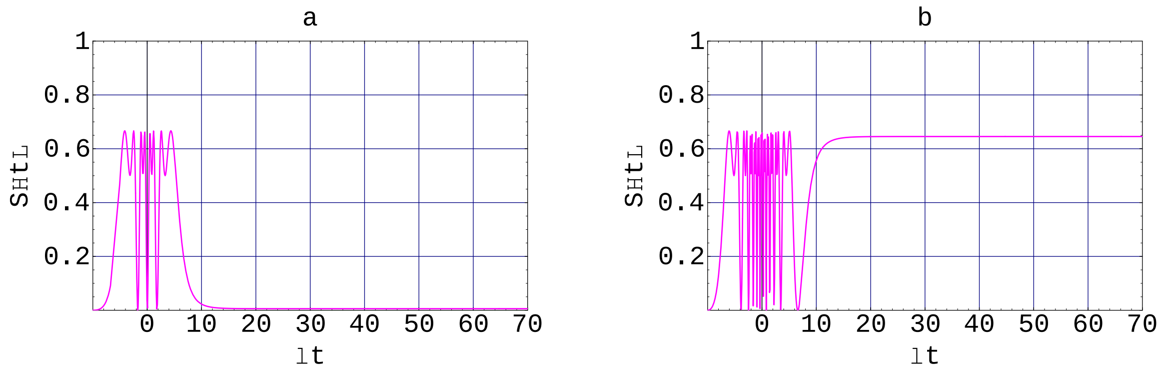

Figure 3: The same as figure 2a but in this figure we consider the modulated function to be time-dependent of the form (12), the initial time is $t_0 = -10\tau$ and different values of n_i, where (a) $n_i = 0$ and (b) $n_i = 15$.

Let us consider the modulated function $\gamma(t)$ to be time-dependent of the form [21, 22]

$$\gamma(t) = \sec\mathrm{h}\left(\frac{t}{2\tau}\right). \tag{12}$$

In this form the coupling increases from a very small value at large negative times to a peak at time $t = 0$, to decrease exponentially at large times. Thus, depending on the value of τ and the initial time t_0, various limits such as adiabatically or rapidly increasing (for $t_0 < \lambda t \leq 0$) or decreasing (for $0 \leq \lambda t < t_0$) coupling can be conveniently studied. This allows us to investigate, analytically, the effect of transients in various different limits of the effect of switching the interaction on and off in the ion-field system. The vanishing of the interaction at large positive times leads to the levelling out of the inversion. It should be noted that the time dependence specified in (12) is one of a class of generalized interactions that offered analytical solutions.

In figure 3 the changes in linear entropy vs the dimensionless quantity λt is plotted when the modulated function is taken to be time-dependent as in equation (12). An intriguing result found in figure 3a, where linear entropy is plotted with an initially Fock state of the fields ($|n_1, n_2\rangle$, with $n_1 = n_2 = 0$) with a time-dependent modulated function, showing clearly the sudden death of entanglement at $\lambda t \simeq 10.2$. A remarkable property of such initial state setting is that entanglement can fall abruptly to zero for a very long time and the entanglement will not be recovered i.e. the state will stay in the disentanglement separable state. On the other hand, we notice that the long-living entanglement can be obtained with large values of the initial Fock state numbers, such as $|n_1 = 15, n_2 = 15\rangle$, (see figure 3b). Therefore, the initial

Fock states placed at this point is a suitable choice to investigate entanglement dynamics for different initial number of photons for the fields. It's not surprising to find that the number of oscillations is increased for higher m_i. At the period $-10 \leq \lambda t \leq 10$, the slight difference lies on the number of oscillations only, while for the later times (say $\lambda t > 10$), the situation becomes completely different.

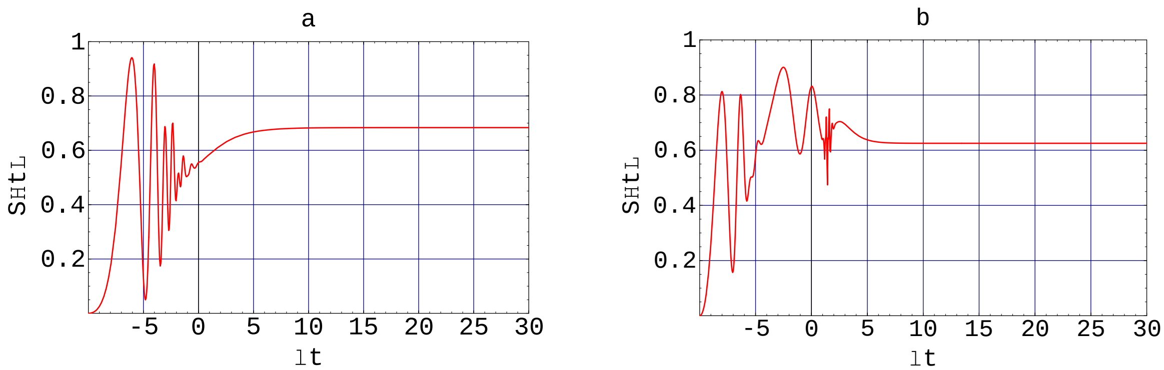

Figure 4: The same as figure 3 but in this figure we consider an initially coherent state field with an average photon number of $\overline{n}_i = 10$ and various value of the number of quanta, where (a) $m_i = 1$ and (b) $m_i = 2$.

The above results and connections are very intriguing, and lead us to ask what is the role played by the initial state in obtaining these associated phenomena of the entanglement. In order to answer to this question, we consider different initial state in figure 4. This figure shows the linear entropy with an initially coherent state field with an average photon number of $\overline{n}_i = 10$ for various number of quanta. The initial time is $t_0 = -10\tau$, so the interaction starts at a fairly low value, peaks and then drops off again. It is interesting to mention here that, as time goes on long-livid entanglement is observed. As a particular but striking enough example we have considered the same value of the system parameters which have been considered in the time-independent case. We have analyzed the long-lived entanglement by considering a multi-photon interaction ($m_1 = m_2 = 2$) in figure 4b. This case is similar to a situation where both $m_1 = m_2 = 1$, because both linear entropies in figure 4a and 4b rise and lower together although they are not equal. The only difference between the two cases is that the maximum long-living entanglement for one photon case is $S(t) \simeq 0.69$, but in the two-photon case is $S(t) \simeq 0.62$. Also, figure 4b demonstrates that the linear entropy peaks show a lowering of the local maximum at the interaction period $-10 \leq \lambda t \leq 0$.

All these results confirm the possibility of a practical observation of time-dependence of the modulated function effects for creating sudden death or long-lived entanglement. Based on

such sensitivity and some other evidence, we suspect that the analytical results presented here, could be attained for different configurations of the three-level systems.

4 Conclusions

Summarizing, we have investigated the dynamics of quantum entanglement for a trapped ion-laser field interaction. An explicit expression is given for a time-independent case and compared with previous studies. Through a three-level trapped ion system we have shown that the commonly assumed initial state setting may affect entanglement in a very different manner. This study reveals that the time-dependent modulated function can be used for generating either entanglement sudden death or long-lived entanglement depending on a proper manipulation of the initial state setting. We hope the presented results can be useful for the ongoing theoretical and experimental efforts in multi-levels particles interaction. Hence, despite the considerable progresses on which we have reported here, a panoply of challenging open questions awaits solution, what simply reflects the decoherence effect as well as the cavity decay or atomic decay.

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References

[1] C. H. Bennett, H. J. Bernstein, S. Popescu and B. Schumacher, Phys. Rev. A 53, 2046 (1996).

[2] T. Yu and J. H. Eberly, Opt. Commun. 264, 393 (2006)

[3] M. Yonac, T. Yu and J. H. Eberly, J. Phys. B: At. Mol. Opt. Phys. 39, S621(2006)

[4] D. Leibfried, R. Blatt, C. Monroe and D. Wineland, Rev. Mod. Phys. 75, 281 (2003)

[5] M. Abdel-Aty, Appl. Phys. B: Laser and Opt. 84, 471 (2006)

[6] D. Leibfried, D. M. Meekhof, B.E. King, C. Monroe, W.M. Itano and D.J. Wineland, J. Mod. Opt. 44, 2485 (1997)

[7] C. Pineda and T. H. Seligman, Phys. Rev. A 73 (2006) 012305.

[8] T. Yu and J. H. Eberly, Phys. Rev. B 66, 193306 (2002)

[9] T. Yu and J. H. Eberly, Phys. Rev. B 68, 165322 (2003)

[10] T. Yu and J. H. Eberly, Phys. Rev. Lett. 93, 140404 (2004).

[11] A. Messina, S. Maniscalco and A. Napoli, J. Mod. Opt. 50, 1 (2003);

[12] H.-I. Yoo and J. H. Eberly, Phys. Rep. 118, 239 (1985); B. W. Shore and P. L. Knight, J. Mod. Opt. 40, 1195 (1993)

[13] H. Moya-Cessa, A. Vidiella-Barranco, J.A. Roversi, and S. M. Dutra, J. Opt. B: Quantum Semiclassical Opt. 2, 21 (2000); H. Moya-Cessa, D. Jonathan and P. L. Knight, J. Mod. Opt. **50**, 265 (2003); W. Vogel and S. Wallentowitz, Manipulation of the quantum state of a trapped ion, in Coherence and statistics of photons and atoms, edited by Jan Perina (Wiley, New York, 2001).

[14] C. A. Blockley, D. F. Walls and H. Risken, Europhys. Lett. 17, 509 (1992); S. S. Sharma and N. K. Sharma, J. Phys. B **35**, 1643 (2002).

[15] V. Vedral, M. B. Plenio and P. L. Knight, "The Physics of Quantum Information", edited by D Bouwmeester, A Ekert and A Zeilinger, Springer (2000); V. Vedral, M. B. Plenio, K. Jacobs, and P. L. Knight, Phys. Rev. A **56**, 4452 (1997).

[16] S. J. D. Phoenix and P. L. Knight, Ann. Phys. (N. Y) **186**, 381 (1988)

[17] S. J. D. Phoenix and P. L. Knight, Phys. Rev. A **44**, 6023 (1991)

[18] S. J. D. Phoenix and P. L. Knight, Phys. Rev. Lett. **66**, 2833 (1991).

[19] D. W. Berry and B. C. Sanders, J. Phys. A 36, 12255 (2003) .

[20] Leibfried D, Meekhof D M, King B E, Monroe C, Itano W M and Winel D J 1996 Phys. Rev. Lett. 77 4281; Turchette Q A et al 2000 Phys. Rev. A 61 063418

[21] S. V. Prants and L. S. Yacoupova, J. Mod. Opt. 39, 961 (1992); A. Joshi and S.V. Lawande, Phys. Rev. A 48, 276 (1993); A. Joshi and S.V. Lawande, Phys. Lett. A 184, 390 (1994).

[22] A. Dasgupta, J. Opt. B: Quantum Semiclass. Opt. 1, 14 (1999).